\# U prostoru $\mathbb{R}^5$ zadan je podprostor $\mathcal{M}$ razapet (generisan) vektorima $(0,0,1,0,0)^T$ i $(0,1,0,1,0)^T$ i podprostor

$$\mathcal{L} = \left\{ (x_1,x_2,x_3,x_4,x_5)^T \in \mathbb{R}^5 \mid x_1 - x_2 + x_3 = 0,\ 2x_1 - 2x_2 + x_3 + x_4 = 0 \right\}$$

(a) Odrediti bazu i dimenziju vektorskih prostora $\mathcal{M}$ i $\mathcal{L}$.

(b) Odrediti dimenziju vektorskog prostora $\mathcal{M} \cap \mathcal{L}$,

(c) Odrediti neku bazu za (direktni) kompliment prostora $\mathcal{L}$ (koji nije ortogonalni komplement).

Rj.

Red vektor $(x_1,x_2,x_3,x_4,x_5)^T$ ćemo u rješenju pisati kao kolona vektor $\begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$. Prema postavci zadatka imamo

$$\mathcal{M} = \operatorname{span}\left\{ \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix} \right\}$$

Dalje

$$\mathcal{L} = \left\{ \begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} \in \mathbb{R}^5 \;\middle|\; \underbrace{\begin{pmatrix} 1 & -1 & 1 & 0 & 0\\ 2 & -2 & 1 & 1 & 0\end{pmatrix}}_{=A} \begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \right\} = \ker A$$

Generator skup za $\ker A$ su vektori iz opšteg rješenja sistema $Ax=0$

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 0\\ 2 & -2 & 1 & 1 & 0\end{pmatrix} \overset{II_v + I_v\cdot(-2)}{\sim} \begin{pmatrix} 1 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\end{pmatrix} \Rightarrow \operatorname{rang} A = 2 = \operatorname{rang} \bar{A}$$

Sistem Ax ima ∞ mnogo rješenja i 3 promjenjive uzimamo proizvoljno npr. $x_2 = s$, $x_4 = t$, $x_5 = u$

$x_1 = x_2 - x_3 \Rightarrow x_1 = s - t$

$-x_3 = -x_4 \Rightarrow x_3 = t$

$$\begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix} s-t\\s\\t\\t\\u\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}s + \begin{pmatrix}-1\\0\\1\\1\\0\end{pmatrix}t + \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}u$$

Time je $\mathcal{L} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Prema tome $\dim \mathcal{M} = 2$, $\dim \mathcal{L} = 3$ a baze za $\mathcal{M}$ i $\mathcal{L}$ su redom

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \quad i \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

b) Prisjetimo se:

<u>Dimenzija sume</u>

<u>Ako su $\mathcal{X}$ i $\mathcal{Y}$ podprostori vektorskog prostora V, tada</u>

$$\underline{\dim(\mathcal{X} + \mathcal{Y}) = \dim(\mathcal{X}) + \dim \mathcal{Y} - \dim(\mathcal{X} \cap \mathcal{Y})}$$

<u>gdje je $\mathcal{X} + \mathcal{Y} = \{ x + y \mid x \in \mathcal{X} \text{ i } y \in \mathcal{Y} \}$.</u>

Ako sa $\mathcal{B}_\mathcal{M}$ označimo bazu za $\mathcal{M}$ a sa $\mathcal{B}_\mathcal{L}$ označimo bazu za $\mathcal{L}$, vidimo da $\mathcal{B}_\mathcal{M} \cup \mathcal{B}_\mathcal{L}$ generira $\mathcal{M} + \mathcal{L}$.

Dimenziju za $\mathcal{M} + \mathcal{L}$ nije teško odrediti, promatramo broj linearno nezavisnih kolona iz $\mathcal{B}_\mathcal{M} \cup \mathcal{B}_\mathcal{L}$:

$$D = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{I_k \leftrightarrow III_k}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{II_v + I_v \cdot (-1)}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{IV_v + II_v \cdot (-1)}{\sim}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{rang } D = 4 \implies \dim(\mathcal{X} + \mathcal{Y}) = 4$$

$$\dim(\mathcal{X}+\mathcal{Y}) = \underset{2}{\dim \mathcal{X}} + \underset{3}{\dim \mathcal{Y}} - \dim(\mathcal{X} \cap \mathcal{Y})$$

$$\dim(\mathcal{X} \cap \mathcal{Y}) = 1$$

U zadatku se ne traži da odredimo bazu za $\mathcal{X}\cap\mathcal{Y}$. Međutim, ako bi željeli da odredimo ovu bazu prvo primjetimo da je

$$\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}+\begin{pmatrix}-1\\0\\1\\1\\0\end{pmatrix}$$

i kako je $\dim(\mathcal{X}\cap\mathcal{Y})=1$ to je $\mathcal{X}\cap\mathcal{Y}=\operatorname{span}\left\{\begin{pmatrix}0\\1\\1\\1\\0\end{pmatrix}\right\}$.

c) Prisjetimo se

Komplementarni podprostori

Za podprostore $\mathcal{X}$ i $\mathcal{Y}$ prostora $\mathcal{V}$ kažemo da su komplementarni kadgod je

$$\mathcal{V}=\mathcal{X}+\mathcal{Y} \quad i \quad \mathcal{X}\cap\mathcal{Y}=0$$

i u ovom slučaju za $\mathcal{V}$ kažemo da je direktna suma od $\mathcal{X}$ i $\mathcal{Y}$, i ovo označavamo sa $\mathcal{V}=\mathcal{X}\oplus\mathcal{Y}$.

Ako su $\mathcal{B}_\mathcal{X}$ i $\mathcal{B}_\mathcal{Y}$ baze za $\mathcal{X}$ i $\mathcal{Y}$ tada

$\mathcal{V}=\mathcal{X}\oplus\mathcal{Y}$ akko $\forall v\in\mathcal{V}\ \exists! x\in\mathcal{X}, y\in\mathcal{Y}$ t.d. $v=x+y$ akko $\mathcal{B}_\mathcal{X}\cap\mathcal{B}_\mathcal{Y}=\emptyset$ i $\mathcal{B}_\mathcal{X}\cup\mathcal{B}_\mathcal{Y}$ je baza za $\mathcal{V}$

Ako sa B označimo matricu

$$B=\begin{pmatrix}1&1&0&0&0\\-1&0&1&1&0\\0&0&0&0&1\end{pmatrix}$$

tada je

$$\operatorname{im}(B^T)=\operatorname{span}\left\{\begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}=\mathcal{L}$$

pa produžimo bazu od $\mathcal{L}$ do baze prostora $\mathbb{R}^5$.

Znamo da je, za proizvoljne matrice A i B

$$\underline{im(A^T) = im(B^T) \text{ akko } A \overset{red}{\sim} B}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \underset{IV_v + I_v \cdot (-1)}{\overset{II_v + I_v}{\sim}} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \underset{V_v + II_v \cdot (-1)}{\overset{IV_v + II_v}{\sim}} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Baza za direktni komplement prostora $\mathcal{L}$ je

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Odrediti sve podprostore od $\mathbb{R}^2$ koji su invarijantni u odnosu na $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.

Rj. Podprostori od $\mathbb{R}^2$ mogu biti dimenzije 0, 1 i 2.

- Trivijalni podprostor $\{\mathbf{0}\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ je jedini nula-dimenzionalni prostor pa je on i jedini nula-invarijantan podprostor od $\mathbb{R}^2$.

Podprostor od $\mathbb{R}^2$ koji je dimenzije 2 mora biti sam $\mathbb{R}^2$ (zašto?). Pa je $\mathbb{R}^2$ jedini dvo-dimenzionalni invarijantan podprostor.

Pravi problem predstavlja pronaći sve jedno-dimenzionalne invarijantne podprostore.

Posmatrajmo jednodimenzionalan podprostor $\mathcal{M}$ koji je generisan sa $x \neq 0$ $(\mathcal{M} = \operatorname{span}\{x\} = \{\alpha x \mid \alpha \in \mathbb{R}\})$ takav da je $A(\mathcal{M}) \subseteq \mathcal{M}$. Tada

$$Ax \in \mathcal{M} \Rightarrow \exists \text{ skalar } \lambda \text{ takav da } Ax = \lambda x \Rightarrow (A - \lambda I)x = 0.$$

Drugim riječima $\mathcal{M} \subseteq \ker(A - \lambda I)$. Kako je $\dim \mathcal{M} = 1$, mora biti slučaj da $\ker(A - \lambda I) \neq 0$ i λ mora biti skalar takav da je $(A - \lambda I)$ singularna matrica.

Pitanje: Zašto $(A - \lambda I)$ ne smije biti nesingularna matrica?

$$A-\lambda I=\begin{pmatrix}-\lambda & 1\\ -2 & 3-\lambda\end{pmatrix}\overset{I_v\leftrightarrow II_v}{\sim}\begin{pmatrix}-2 & 3-\lambda\\ -\lambda & 1\end{pmatrix}\overset{II_v+I_v\cdot\frac{-\lambda}{2}}{\sim}\begin{pmatrix}-2 & 3-\lambda\\ 0 & 1+\frac{\lambda(\lambda-3)}{2}\end{pmatrix}$$

A odavde vidimo da će $A-\lambda I$ biti singularna matrica akko $1+\frac{\lambda^2-3\lambda}{2}=0$ tj. akko je λ korijen od

$$\lambda^2-3\lambda+2=0.$$

Prema tome $\lambda=1$ i $\lambda=2$ i direktno računanje povlači dva jedno-dimenzionalna invarijantna podprostora

$$M_1=\ker(A-I)=\ker\begin{pmatrix}-1 & 1\\ -2 & 2\end{pmatrix}=\operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$$

i $M_2=\ker(A-2I)=\operatorname{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ i ovo su tražena rješenja

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Usput, primjetimo da $\mathcal{B}=\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right\}$ je baza za $\mathbb{R}^2$

i $[A]_{\mathcal{B}}=Q^{-1}AQ=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ gdje $Q=\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$

U opštem slučaju, skalare λ za koje $(A-\lambda I)$ je singularna zovemo svojstvene vrijednosti od A, i nenula vektore u $\ker(A-\lambda I)$ su poznati kao svojstveni vektori za A. Kao što ovaj primjer pokazuje, svojstvene vrijednosti i svojstveni vektori su od velike važnosti u identificiranju invarijantnih podprostora i u svođenju matrica pomoću transformacija sličnosti.

(#) Izračunati 1-, 2- i ∞-norme vektora $x=\begin{pmatrix}2\\1\\-4\\-2\end{pmatrix}$ i $y=\begin{pmatrix}1+i\\1-i\\1\\4i\end{pmatrix}$.

R.:

$$\|x\|_p \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|$$

Za $x=\begin{pmatrix}2\\1\\-4\\-2\end{pmatrix}$ imamo

$$\|x\|_2 = (4+1+16+4)^{\frac{1}{2}} = \sqrt{25} = 5$$

$$\|x\|_1 = 2+1+4+2 = 9$$

$$\|x\|_\infty = \max\{2,1,4,2\} = 4$$

Za $y=\begin{pmatrix}1+i\\1-i\\1\\4i\end{pmatrix}$ imamo

$$|1+i| = \sqrt{1+1} = \sqrt{2}$$

$$|1-i| = \sqrt{1+1} = \sqrt{2}$$

$$|4i| = \sqrt{16+0} = 4$$

$$\|y\|_2 = (2+2+1+16)^{\frac{1}{2}} = \sqrt{21}$$

$$\|y\|_1 = \sqrt{2}+\sqrt{2}+1+4 = 2\sqrt{2}+5$$

$$\|y\|_\infty = \max\{\sqrt{2},\sqrt{2},1,4\} = 4$$

Primjetimo da je

$$\|x\|_\infty < \|x\|_2 < \|x\|_1$$

$$\|y\|_\infty < \|y\|_2 < \|y\|_1$$

\# Ako je $x, y \in \mathbb{R}^n$ tako da $\|x-y\|_2 = \|x+y\|_2$, šta je $x^T y$?

Rj.:

$$\|x-y\|_2 = \|x+y\|_2 \quad /^2$$

$$\|x-y\|_2^2 = \|x+y\|_2^2$$

$$(x-y)^T(x-y) = (x+y)^T(x+y)$$

$$(x^T-y^T)(x-y) = (x^T+y^T)(x+y)$$

$$x^Tx - x^Ty - y^Tx - y^Ty = x^Tx + x^Ty + y^Tx + y^Ty$$

$$\|x\|_2^2 - x^Ty - x^Ty - \|y\|_2^2 = \|x\|_2^2 + x^Ty + x^Ty + \|y\|_2^2$$

$$-2x^Ty = 2x^Ty$$

$$4x^Ty = 0$$

$$x^Ty = 0$$

(#) Dat je unitarni prostor $\mathcal{P}_3$, polinoma stepena ≤ 3, sa skalarnim (unutrašnjim) proizvodom

$$\langle p, q \rangle = \frac{1}{4} \sum_{i=0}^{3} p(\lambda_i) q(\lambda_i)$$

gdje su $\lambda_0 = 3$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = -3$. Primjenom Gram-Schmidtovog procesa ortonormirati bazu $\{-1, x, -x^2, x^3\}$ i dobiti polinome $\{P_1(x), P_2(x), P_3(x), P_4(x)\}$ koji su ortogonalni i za koje vrijedi da je $\|P_i\|^2 = P_i(\lambda_0)$, za $\forall i = 1,2,3,4$.

Rj: Prisjetimo se Gram-Šmitove ortogonalne procedure:
Ako je $\mathcal{B} = \{x_1, x_2, \dots, x_n\}$ baza za unitarni prostor $\mathcal{S}$, tada Gram-Šmitov niz definisan sa

$$u_1 = \frac{x_1}{\|x_1\|} \quad i \quad u_k = \frac{x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i}{\|x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i\|} \quad \text{za } k = 2, 3, \dots, n$$

je ortonormirana baza za $\mathcal{S}$.

Algoritam

za $k=1$: $\quad u_1 \leftarrow \dfrac{x_1}{\|x_1\|}$

za $k>1$: $\quad u_k \leftarrow x_k - \sum_{i=1}^{k-1} \langle u_i, x_k \rangle u_i$

$$u_k \leftarrow \frac{u_k}{\|u_k\|}$$

Pa slijedimo ovaj algoritam i formirajmo prvo ortonormirane polinome r_0, r_1, r_2 i r_3.

Dat je skup $\{-1, x, -x^2, x^3\}$
$$g_1 \quad g_2 \quad g_3 \quad g_4$$

$k=1$: $\quad r_1 \leftarrow \frac{g_1}{\|g_1\|}$

Kako je $\langle p, g\rangle = \frac{1}{4}\left(p(3)g(3) + p(1)g(1) + p(-1)g(-1) + p(-3)g(-3)\right)$

to je

$$\|g_1\|^2 = \langle g_1, g_1\rangle = \frac{1}{4}(1+1+1+1) = \frac{1}{4}\cdot 4 = 1$$

$$\|g_1\| = \sqrt{1} = 1$$

$$r_1 \leftarrow \frac{-1}{1} = -1 \qquad r_1(x) = -1$$

$k=2$: $\quad r_2 \leftarrow g_2 - \langle r_1, g_2\rangle r_1, \qquad r_2 \leftarrow \frac{r_2}{\|r_2\|}$

$$\langle r_1, g_2\rangle = \frac{1}{4}\left((-1)\overset{-3}{3} + (-1)\cdot \overset{-1}{1} + (-1)\cdot\overset{1}{(-1)} + (-1)\cdot\overset{3}{(-3)}\right) = 0$$

$$r_2 \leftarrow x - 0 = x$$

$$\|x\|^2 = \langle x, x\rangle = \frac{1}{4}(9+1+1+9) = 5 \quad \Rightarrow \quad \|x\| = \sqrt{5}$$

$$r_2 \leftarrow \frac{x}{\sqrt{5}} \qquad r_2(x) = \frac{1}{\sqrt{5}}x$$

$3, 1, -1, -3$

$r_1(x) = -1$

$r_2(x) = \frac{1}{\sqrt{5}}x$

$g_3(x) = -x^2$

$k=3$: $\quad r_3 \leftarrow g_3 - \langle r_1, g_3\rangle r_1 - \langle r_2, g_3\rangle r_2$

$$\langle r_1, g_3\rangle = \frac{1}{4}(9+1+1+9) = 5$$

$$\langle r_2, g_3\rangle = \frac{1}{4}\left(-\frac{27}{\sqrt{5}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} + \frac{27}{\sqrt{5}}\right) = 0$$

$$r_3 \leftarrow -x^2 + 5, \qquad \|-x^2+5\|^2 = \langle -x^2+5, -x^2+5\rangle = \frac{1}{4}(16+16+16+16) = 16$$

$\| -x^2+5 \| = 4$

$r_3 \leftarrow \frac{1}{4}(-x^2+5) = -\frac{1}{4}x^2 + \frac{5}{4}$ $\qquad r_3(x) = -\frac{1}{4}x^2 + \frac{5}{4}$

Za sad imamo

$r_1(x) = -1, \quad r_2(x) = \frac{1}{\sqrt{5}}x, \quad r_3(x) = -\frac{1}{4}x^2 + \frac{5}{4}, \quad g_4 = x^3$

$k=4$:
$r_4 \leftarrow g_4 - \langle r_1, g_4 \rangle r_1 - \langle r_2, g_4 \rangle r_2 - \langle r_3, g_4 \rangle r_3$

$r_1 \cdot g_4 = -x^3$

$\langle r_1, g_4 \rangle = \frac{1}{4}(-27-1+1+27) = 0$

$r_2 \cdot g_4 = \frac{1}{\sqrt{5}}x^4$ $\qquad 164:4=41$

$\langle r_2, g_4 \rangle = \frac{1}{4}\left(\frac{81}{\sqrt{5}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} + \frac{81}{\sqrt{5}}\right) = \frac{41}{\sqrt{5}}$

$r_3 \cdot g_4 = -\frac{1}{4}x^5 + \frac{5}{4}x^3$

$\langle r_3, g_4 \rangle = \frac{1}{4}\left(-\frac{1}{4}3^5 + \frac{5}{4}3^3 - \frac{1}{4} + \frac{5}{4} + \frac{1}{4} - \frac{5}{4} + \frac{1}{4}3^5 - \frac{5}{4}3^5\right) = 0$

$r_4 \leftarrow x^3 - \frac{41}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}}x = x^3 - \frac{41}{5}x$

$\| x^3 - \frac{41}{5}x \|^2 = \langle x^3 - \frac{41}{5}x, x^3 - \frac{41}{5}x \rangle = \ldots = \frac{144}{5}$

$\| x^3 - \frac{41}{5}x \| = \frac{12}{\sqrt{5}}, \quad \frac{1}{\| x^3 - \frac{41}{5}x \|} = \frac{\sqrt{5}}{12}$

$r_4(x) = \frac{\sqrt{5}}{12}x^3 - \frac{41\sqrt{5}}{60}x$

Skup $\{ r_1(x) = -1, \ r_2(x) = \frac{1}{\sqrt{5}}x, \ r_3(x) = -\frac{1}{4}x^2 + \frac{5}{4}, \ r_4(x) = \frac{\sqrt{5}}{12}x^3 - \frac{41\sqrt{5}}{60}x \}$
je ortonormirana baza prostora $\mathcal{P}_3$ u odnosu na dati skalarni proizvod.

Ostalo je još da formiramo polinome $\{p_1(x), p_2(x), p_3(x), p_4(x)\}$ koji su ortogonalni i za koje vrijedi $\|p_i\|^2 = p_i(\lambda_0)$.

Primjetimo da za proizvoljne realne brojeve $\alpha_1, \alpha_2, \alpha_3$ i α_4 skup $\{\alpha_1 r_1, \alpha_2 r_2, \alpha_3 r_3, \alpha_4 r_4\}$ i dalje formira ortogonalan sistem.

Sad imamo

$$\left.\begin{array}{l} r_1(x) = -1 \\ \|r_1\| = 1 \end{array}\right\} \Rightarrow \quad p_1(x) = 1 \quad \left(1 = \|p_1\|^2 = p_1(3) = 1\right)$$

$$(p_1(x) = (-1)\cdot r_1)$$

$$r_2(x) = \frac{1}{\sqrt{5}}x, \quad r_2(3) = \frac{3}{\sqrt{5}} \qquad p_2(x) = \alpha \cdot r_2(x)$$

$$\|r_2\| = 1$$

$$\|p_2\|^2 = \langle p_2, p_2 \rangle = \alpha^2 \langle r_2, r_2 \rangle = \alpha^2 \|r_2\|^2 = \alpha^2$$

$$p_2(\lambda_0) = p_2(3) = \alpha \cdot r_2(3) = \alpha \frac{3}{\sqrt{5}}$$

$$\alpha^2 = \frac{3}{\sqrt{5}}\alpha \Rightarrow \alpha = \frac{3}{\sqrt{5}}$$

$$p_2(x) = \frac{3}{5}x \qquad \left(\frac{9}{5} = \|p_2\| = p_2(\lambda_0) = \frac{9}{5}\right)$$

$$r_3 = -\frac{1}{4}x^2 + \frac{5}{4} \qquad p_3(x) = \beta r_3(x)$$

$$\|r_3\| = 1$$

$$\|p_3\|^2 = \langle \beta r_3, \beta r_3 \rangle = \beta^2 \langle r_3, r_3 \rangle = \beta^2$$

$$p_3(\lambda_0) = p_3(3) = \beta \cdot \left(-\frac{1}{4}3^2 + \frac{5}{4}\right) = \beta\left(-\frac{4}{4}\right) = -\beta$$

$$\beta = -1$$

$$p_3(x) = \frac{1}{4}x^2 - \frac{5}{4} \qquad \left(p_3(\lambda_0) = 1, \ \|p_3\|^2 = 1\right)$$

$$r_4(x) = \frac{\sqrt{5}}{12}x^3 - \frac{41\sqrt{5}}{60}x \qquad p_4(x) = \gamma \cdot r_4(x), \qquad p_4(\lambda_0) = \frac{27\sqrt{5}\cdot 5 - 41\sqrt{5}\cdot 3}{60}\gamma = \frac{12\sqrt{5}}{60}\gamma = \frac{\sqrt{5}}{5}\gamma$$

$$p_4(x) = \frac{1}{12}x^3 - \frac{41}{60}x$$

Traženi skup ortogonalnih polinoma za koje vrijedi $\|p_i\|^2 = p_i(\lambda_0)$ je

$$\left\{p_1(x) = 1, \ p_2(x) = \frac{3}{5}x, \ p_3(x) = \frac{1}{4}x^2 - \frac{5}{4}, \ p_4(x) = \frac{1}{12}x^3 - \frac{41}{60}x\right\}.$$